Portfolio Choice (2)

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Outline

Mean-variance analysis:

- Multiple risky assets, no safe asset
  - Lagrangian approach
  - Global minimum-variance portfolio
  - Mutual fund theorem
- Multiple risky assets and one safe asset
- Practical problems
  - Do investors obey the mutual fund theorem?
  - Is mean-variance analysis usable in practice?
  - The need for shortcut approaches

Multiple Risky Assets, No Safe Asset

With two assets, the mean return target uniquely defines the portfolio weights.

This is no longer true when we have N risky assets. Now the problem is to find the "minimum-variance frontier" of portfolios that have minimum variance for a given mean return.

- $\overline{R}$ vector of mean returns,
- Σ variance-covariance matrix of returns,
- w vector of portfolio weights, and
- ι vector of ones.

The Lagrangian Approach

$$\min_w \frac{1}{2} w' \Sigma w \text{ s.t.}$$

$$\begin{aligned} \overline{R}' w &= \overline{R}_p \\ \iota' w &= 1 \, . \end{aligned}$$

Set up the Lagrangian

$$\mathcal{L}(w, \lambda_1, \lambda_2) = \frac{1}{2} w' \Sigma w + \lambda_1 (\overline{R}_p - \overline{R}' w) + \lambda_2 (1 - \iota' w)$$

First-Order Conditions

$$\mathcal{L}(w, \lambda_1, \lambda_2) = \frac{1}{2} w' \Sigma w + \lambda_1 (\overline{R}_p - \overline{R}' w) + \lambda_2 (1 - \iota' w)$$

First-order conditions are

$$\Sigma w = \lambda_1 \overline{R} + \lambda_2 \iota \ .$$

Premultiply both sides by Σ^{-1} to get:

$$w = \lambda_1 \Sigma^{-1} \overline{R} + \lambda_2 \Sigma^{-1} \iota \ .$$

Solving for Lagrange Multipliers

Use the two constraints:

$$
\begin{aligned}
\overline{R}_p &= \overline{R}' w = \lambda_1 \overline{R}' \Sigma^{-1} \overline{R} + \lambda_2 \overline{R}' \Sigma^{-1} \iota = \lambda_1 A + \lambda_2 B \\
1 &= \iota' w = \lambda_1 \iota' \Sigma^{-1} \overline{R} + \lambda_2 \iota' \Sigma^{-1} \iota = \lambda_1 B + \lambda_2 C \ ,
\end{aligned}
$$

where

- $A \equiv \overline{R}' \Sigma^{-1} \overline{R} > 0$
- $B \equiv \overline{R}' \Sigma^{-1} \iota = \iota' \Sigma^{-1} \overline{R}$
- $C \equiv \iota' \Sigma^{-1} \iota > 0.$

Solving these equations, we get

$$
\lambda_1 = \frac{C\overline{R}_p - B}{D} \ , \qquad \lambda_2 = \frac{A - B\overline{R}_p}{D} \ ,
$$

where $D \equiv AC - B^2$.

Minimized Variance

$$
\begin{aligned}
\sigma_p^2 &= w'\Sigma w = w'\Sigma(\lambda_1\Sigma^{-1}\overline{R} + \lambda_2\Sigma^{-1}\iota) \\
&= \lambda_1 w'\overline{R} + \lambda_2 w'\iota = \lambda_1\overline{R}_p + \lambda_2 \\
&= \frac{A - 2B\overline{R}_p + C\overline{R}_p^2}{D}.
\end{aligned}
$$

Thus $d\sigma_p^2/d\overline{R}_p = \lambda_1$. λ_1 measures the variance cost of a higher mean return target, and it is increasing in $\overline{R}_p$.

Global Minimum-Variance Portfolio

Drop the mean constraint, or equivalently set $\lambda_1 = 0$. We get

$$w_G = \lambda_2 \Sigma^{-1} \iota \, ,$$

$$1 = \iota' w_G = \lambda_2 \iota' \Sigma^{-1} \iota \, .$$

So $\lambda_2 = 1/(\iota' \Sigma^{-1} \iota) = 1/C$, and

$$w_G = \frac{\Sigma^{-1} \iota}{\iota' \Sigma^{-1} \iota} \, .$$

Mean GMV Portfolio Return

$$w_G' \overline{R} = \left(\frac{\iota' \Sigma^{-1} \overline{R}}{\iota' \Sigma^{-1} \iota} \right) = \frac{B}{C}.$$

We expect the mean return on the global minimum-variance portfolio to be positive, and thus we expect B to be positive.

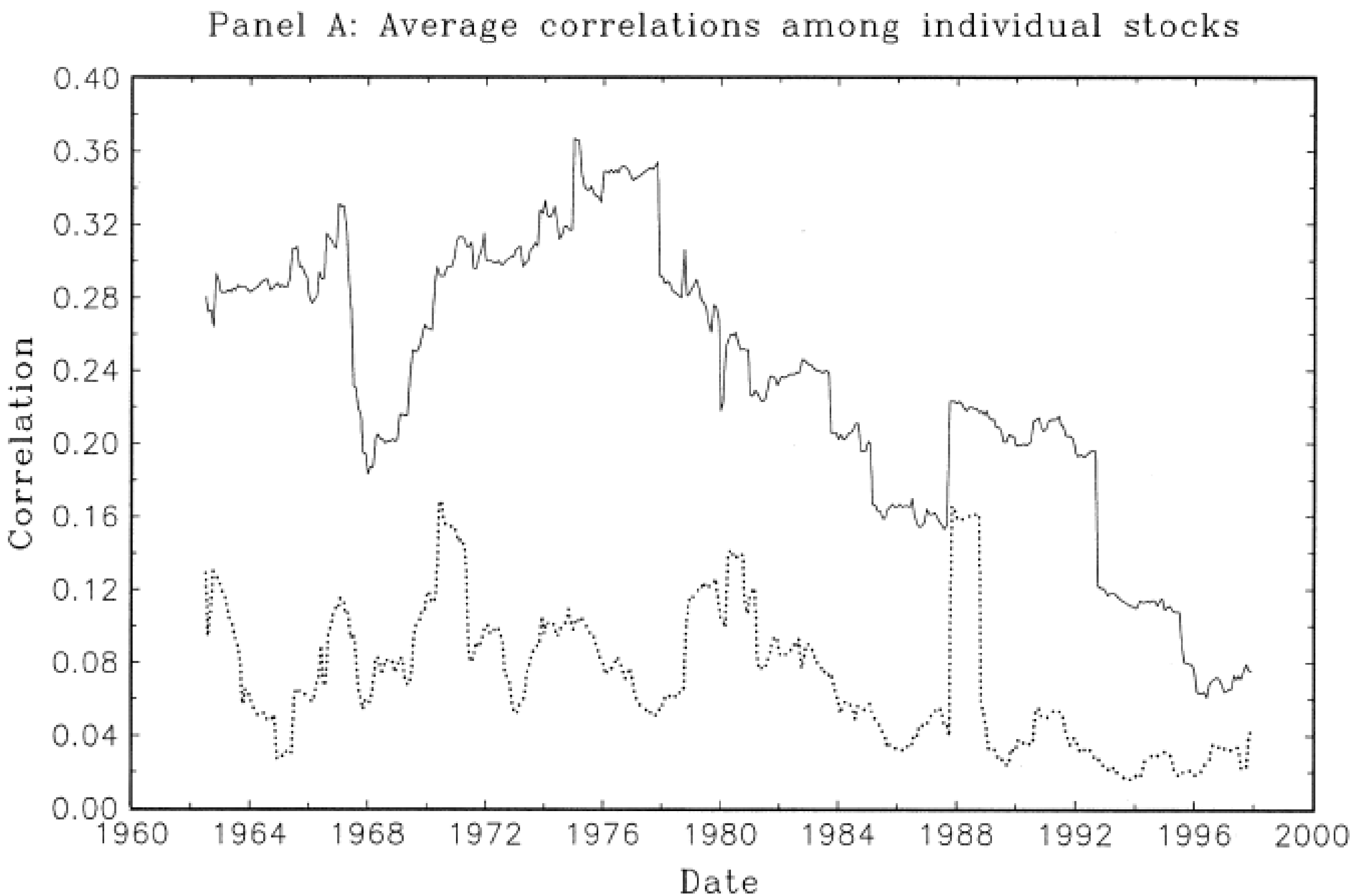
In the general model with an arbitrary mean return constraint, we can verify that when $\overline{R}_p > B/C$, then the Lagrange multiplier for the mean constraint, $\lambda_1 > 0$. The set of minimum-variance portfolios that satisfy this condition is called the *mean-variance efficient set*.

Variance of GMV Portfolio Return

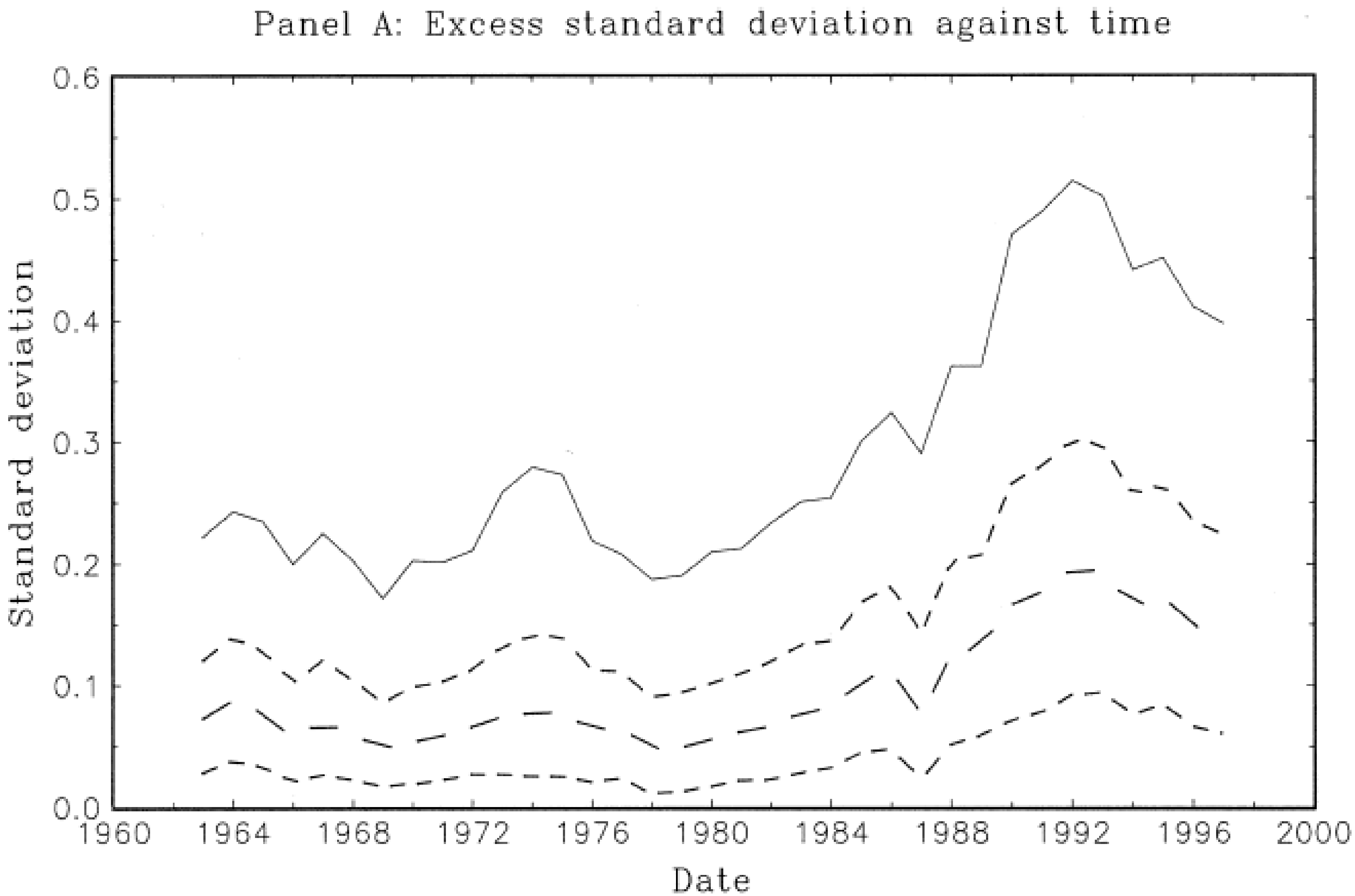
$$w_G'\Sigma w_G = \frac{\iota'\Sigma^{-1}\Sigma\Sigma^{-1}\iota}{(\iota'\Sigma^{-1}\iota)^2} = \frac{1}{(\iota'\Sigma^{-1}\iota)}.$$

This simplifies in the case where all assets are symmetrical, having the *same* variance and the *same* correlation ρ with each other. Then the global minimum-variance portfolio is equally weighted, $w_G = \iota/N$, and

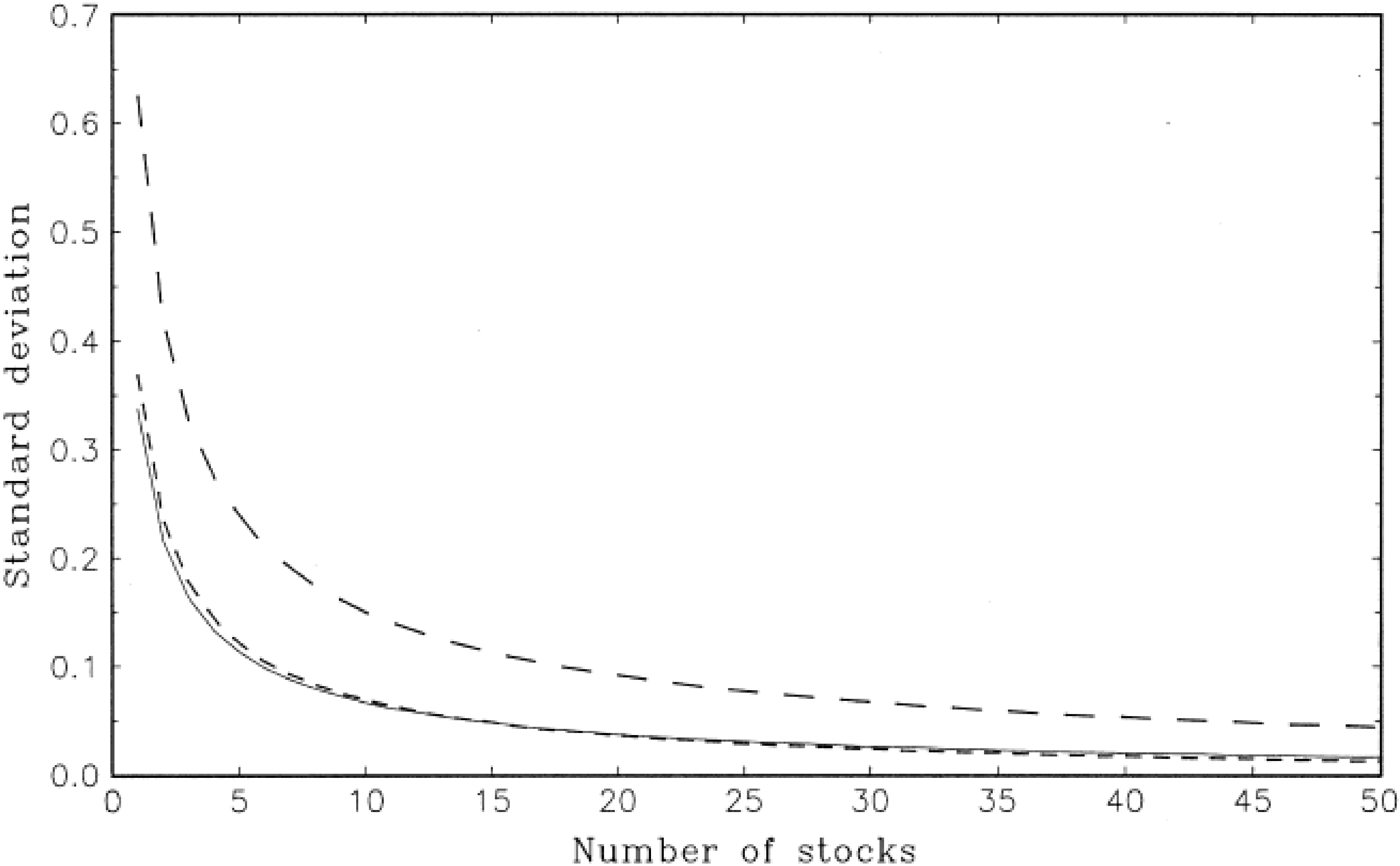
$$\begin{aligned} w_G'\Sigma w_G &= \frac{\iota'\Sigma\iota}{N^2} = \frac{N^2\rho\sigma^2}{N^2} + \frac{N(1-\rho)\sigma^2}{N^2} \\ &= \rho\sigma^2 + \frac{(1-\rho)\sigma^2}{N}. \end{aligned}$$



Campbell, Lettau, Malkiel, and Xu, *Journal of Finance* 2001


Panel A: Excess standard deviation against time
Standard deviation
0.0
0.1
0.2
0.3
0.4
0.5
0.6
1960
1964
1968
1972
1976
1980
1984
1988
1992
1996
2000
Date

Panel B: Excess standard deviation against number of stocks



Mutual Fund Theorem

Rewrite the solution as

$$\begin{aligned} w &= \lambda_1 \iota' \Sigma^{-1} \overline{R} \left(\frac{\Sigma^{-1} \overline{R}}{\iota' \Sigma^{-1} \overline{R}} \right) + \lambda_2 \iota' \Sigma^{-1} \iota \left(\frac{\Sigma^{-1} \iota}{\iota' \Sigma^{-1} \iota} \right) \\ &= \lambda_1 B \left(\frac{\Sigma^{-1} \overline{R}}{\iota' \Sigma^{-1} \overline{R}} \right) + \lambda_2 C w_G , \end{aligned}$$

Since $\lambda_1 B + \lambda_2 C = 1$, the optimal portfolio is a combination of two portfolios, the second of which is the global minimum-variance portfolio, and the first of which invests more heavily in assets with high mean returns.

Tobin (1958): Two mutual funds are enough to meet all investors' needs.

Peter Bernstein: Any other view is the "interior decorator fallacy".

Multiple Risky Assets and One Safe Asset

Write the riskless asset return as R_f. Rewrite the problem as one of choosing weights w in the risky assets, where the portfolio is completed by lending or borrowing at the riskless rate R_f. Thus we no longer require $\iota' w = 1$. Drop this constraint and write the problem as

$$\min_{w} \frac{1}{2} w' \Sigma w \quad \text{s.t.} \quad (\overline{R} - R_f \iota)' w = (\overline{R}_p - R_f) \ .$$

Set up the Lagrangian

$$\mathcal{L}(w_1, w_2, \lambda_1) = \frac{1}{2}(w' \Sigma w) + \lambda_1 (\overline{R}_p - R_f - (\overline{R} - R_f \iota)' w).$$

Solution

$$\frac{\partial \mathcal{L}}{\partial w} = \Sigma w - \lambda_1 (\overline{R} - R_f \iota) = 0$$

so

$$w = \lambda_1 \Sigma^{-1} (\overline{R} - R_f \iota) .$$

Mutual fund theorem: All investors hold a combination of the safe asset and a unique mutual fund containing risky assets (the "tangency portfolio").

The weights are determined by the mean return target:

$$\overline{R}_p - R_f = (\overline{R} - R_f \iota)' w = \lambda_1 (\overline{R} - R_f \iota)' \Sigma^{-1} (\overline{R} - R_f \iota) = \lambda_1 E ,$$

where $E \equiv (\overline{R} - R_f \iota)' \Sigma^{-1} (\overline{R} - R_f \iota)$. Thus

$$\lambda_1 = \frac{\overline{R}_p - R_f}{E} .$$

Solution

Also,

$$\sigma_p^2 = w'\Sigma w = \lambda_1^2(\overline{R} - R_f\iota)'\Sigma^{-1}\Sigma\Sigma^{-1}(\overline{R} - R_f\iota) = \lambda_1^2 E \ .$$

Thus

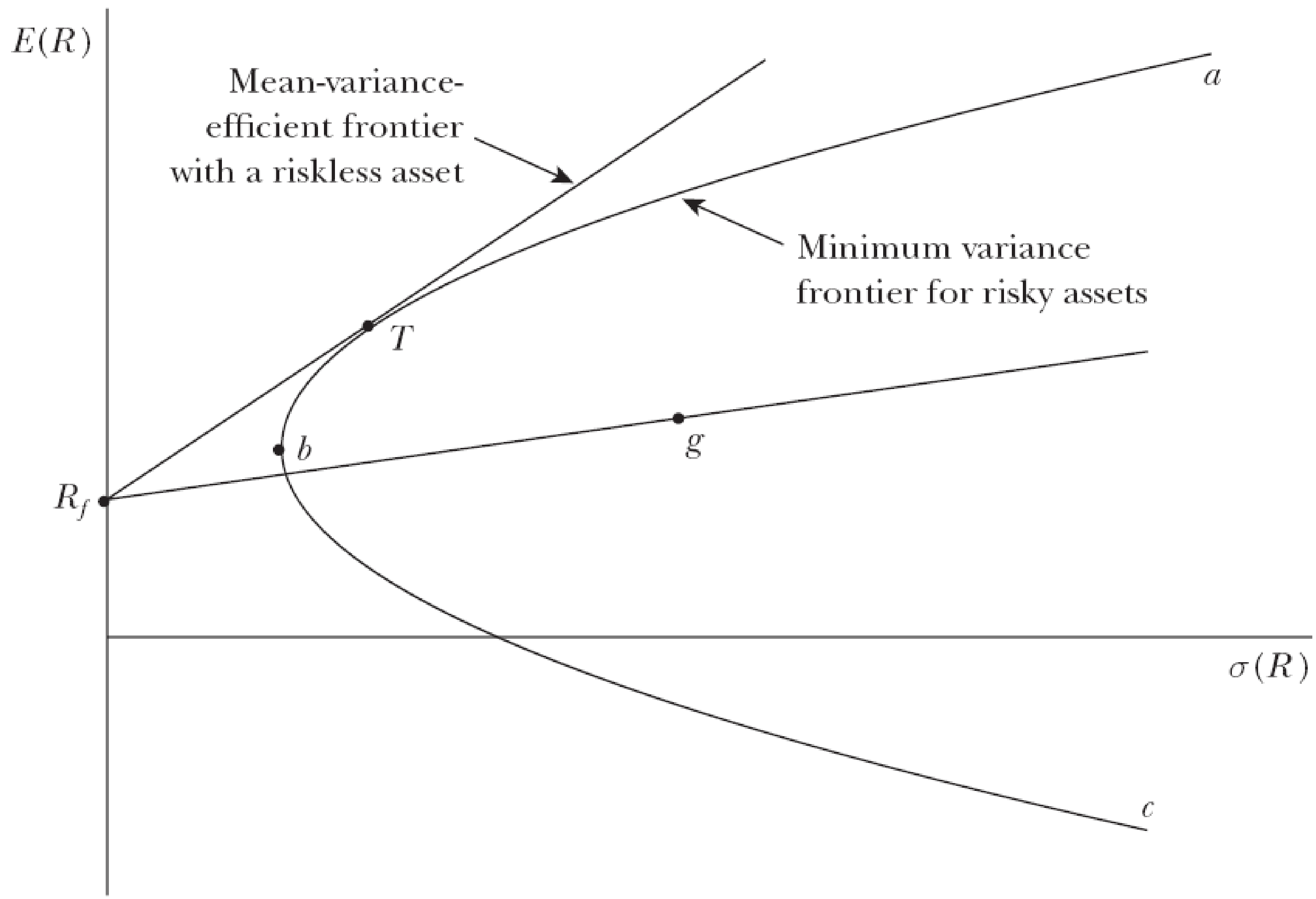
$$\sigma_p^2 = \frac{(\overline{R}_p - R_f)^2}{E} \ ,$$

and

$$\mid \overline{R}_p - R_f \mid = \sqrt{E}\sigma_p \ .$$

The Sharpe ratio of the tangency portfolio is $\sqrt{E}$.

Figure 1
Investment Opportunities



Do Investors Obey the Mutual Fund Theorem?

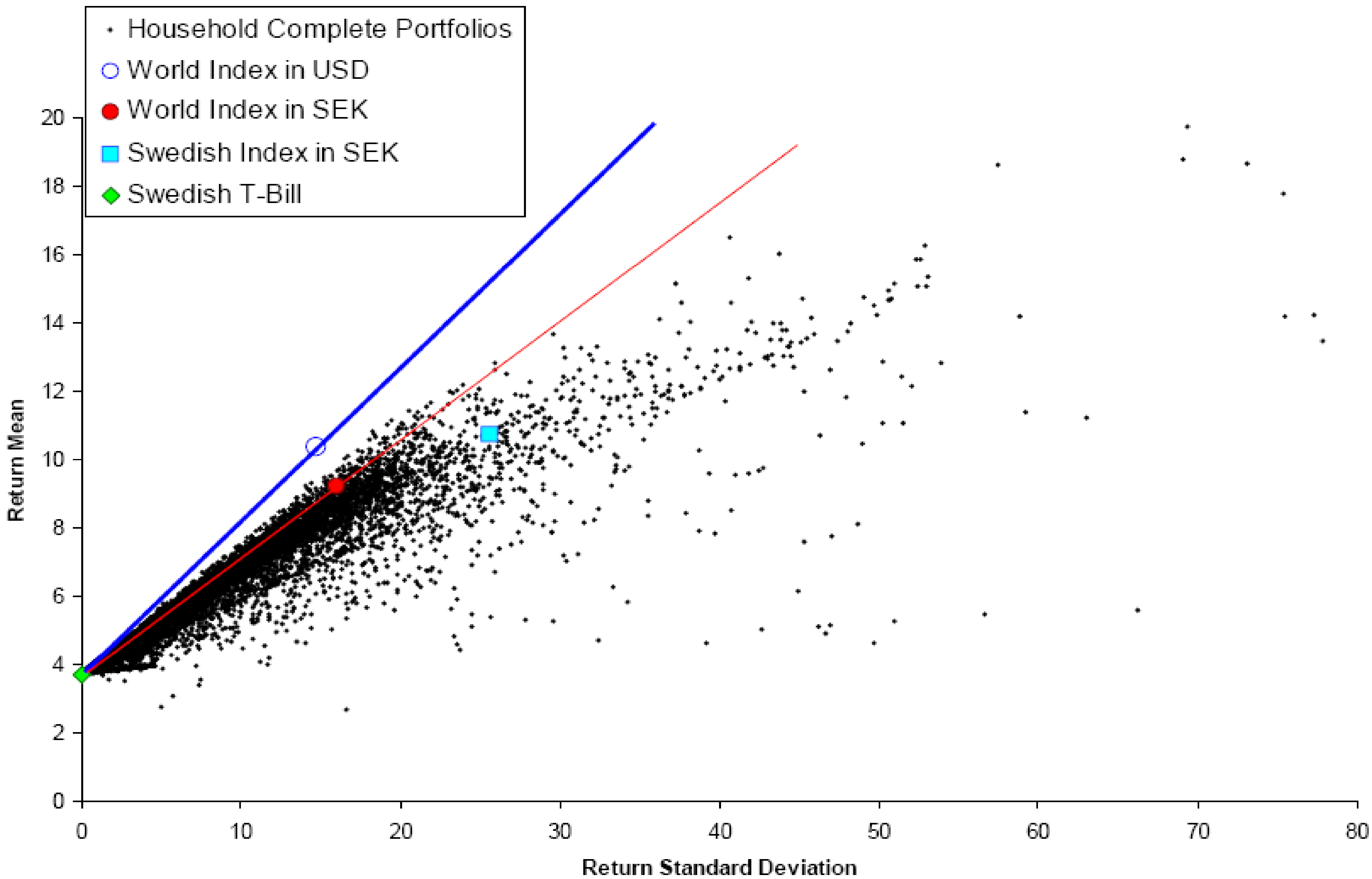
Canner-Mankiw-Weil (1997) "Asset Allocation Puzzle" is that investors tend to shift the composition of the risky portfolio towards safer risky assets when they become more conservative, rather than diluting a given risky portfolio with more cash.

TABLE 1—ASSET ALLOCATIONS RECOMMENDED BY FINANCIAL ADVISORS

	Percent of portfolio			
Advisor and investor type	Cash	Bonds	Stocks	Ratio of bonds to stocks
A. Fidelity[a]				
Conservative	50	30	20	1.50
Moderate	20	40	40	1.00
Aggressive	5	30	65	0.46
B. Merrill Lynch[b]				
Conservative	20	35	45	0.78
Moderate	5	40	55	0.73
Aggressive	5	20	75	0.27
C. Jane Bryant Quinn[c]				
Conservative	50	30	20	1.50
Moderate	10	40	50	0.80
Aggressive	0	0	100	0.00
D. *The New York Times*[d]				
Conservative	20	40	40	1.00
Moderate	10	30	60	0.50
Aggressive	0	20	80	0.25

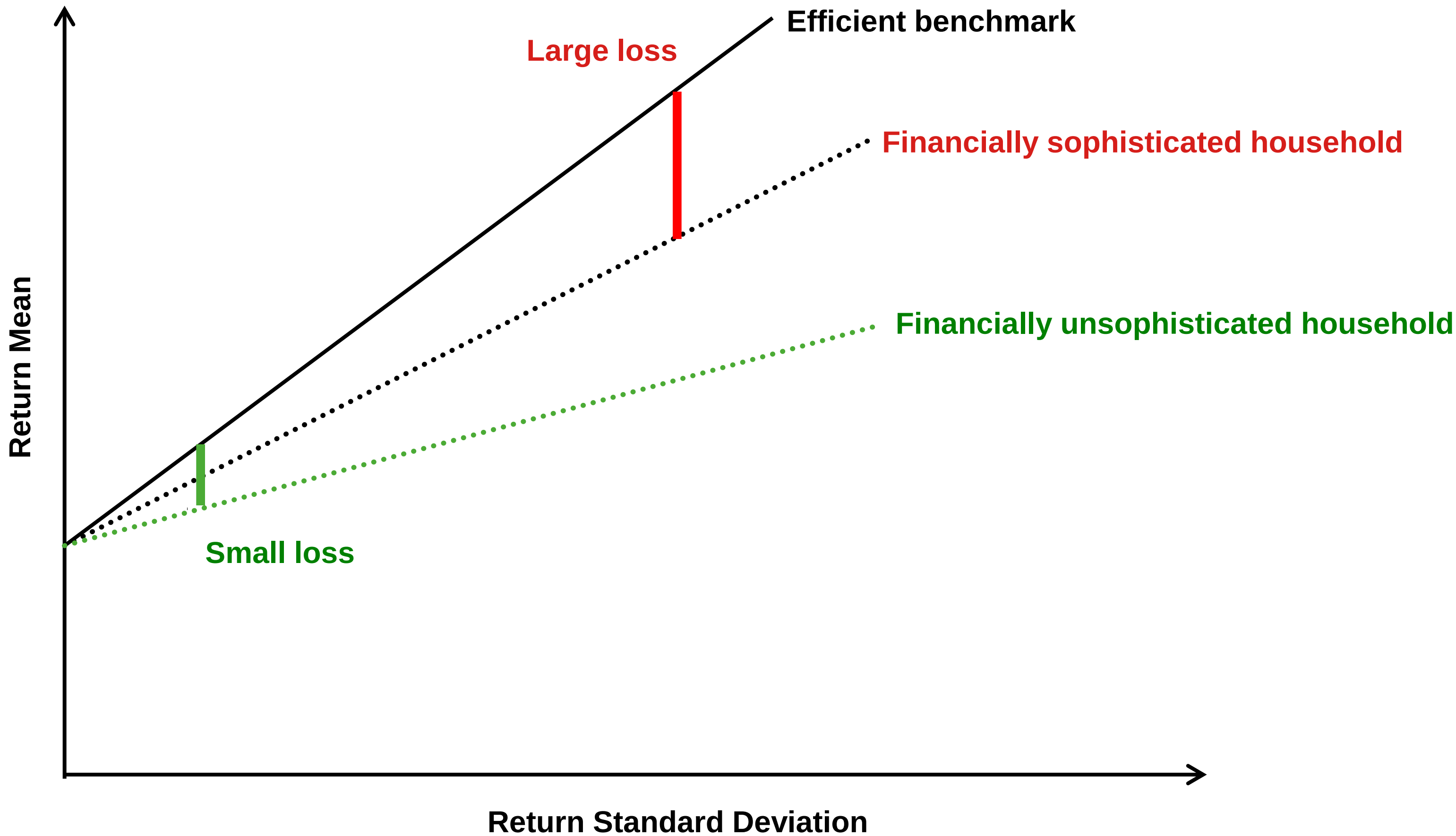
Sources:
[a] Mark, 1993.
[b] Underwood and Brown, 1993.
[c] Quinn, 1991.
[d] Rowland, 1994.

Figure 2B: Complete Portfolios



Campbell, Calvet, and Sodini, *JPE* forthcoming, 2007

FIGURE 3. IMPACT OF FINANCIAL SOPHISTICATION ON RETURN LOSS



The figure illustrates the impact of financial sophistication on the complete return loss. Rich and educated households select portfolios with a high Sharpe ratio but also a high risky share, resulting in a high complete return loss. Conversely, unsophisticated households allocate a small fraction of their financial wealth to an inefficient risky portfolio, and overall incur low complete portfolio return losses.

Is Mean-Variance Analysis Usable in Practice?

- Estimates of means are imprecise over short periods.
  - But means may not be constant over long periods.
- The variance-covariance matrix Σ has $N(N+1)/2$ variances and covariances that have to be estimated. This can be a very large number!
  - If $N \geq T$, the historical variance-covariance matrix is always singular. This means that it cannot be inverted. There will appear to be riskless combinations of risky assets in the data.
  - Even if $N < T$, if N is large the data will suggest that some combinations of risky assets are almost riskless. This can lead to a highly leveraged portfolio.
  - DeMiguel, Garlappi, and Yan (2009) report better out-of-sample properties for an equal-weighted portfolio (a naively diversified portfolio) than an estimated mean-variance optimal portfolio (but this may be due to high return target).

Exhibit 3 Historical Asset Mix

	1992	1996	2000	2004	2008	2010
Domestic Equities	40	36	22	15	12	11
Foreign Equities	18	15	15	10	12	11
Emerging Markets	-	9	9	5	10	11
Private Equities	12	15	15	13	11	13
Total	**70**	**75**	**61**	**43**	**45**	**46**
Absolute Return [a]	-	-	5	12	18	16
High-Yield	2	2	3	5	1	2
Commodities [b]	6	3	6	13	17	14
Real Estate	7	7	7	10	9	9
Total	**15**	**12**	**21**	**40**	**45**	**41**
Domestic Bonds	15	13	10	11	5	4
Foreign Bonds	5	5	4	5	3	2
Inflation-indexed Bonds	-	-	7	6	7	5
Cash	(5)	(5)	(3)	(5)	(5)	2
Total	**15**	**13**	**18**	**17**	**10**	**13**

Source: Company documents.

[a]Includes external managers whose portfolios are invested such that their performance is less sensitive to particular market indices and it can best be evaluated as an "absolute return" rather than some return relative to a market.

[b]Including both commodities and natural resources.

Exhibit 17 Capital Market Assumptions: Real Return, Risk, and Correlations

	Real Return (%)	Risk[a] (%)	Correlations: Dom. Equity	For. Equity	Emer. Markets	Private Equity	Abs. Return	High Yield	Comm.	Nat. Res.	Real Estate	Dom. Bonds	For. Bonds	Infl - Indx.	Cash
Domestic Equity	5.75%	15.5%	1.00	0.85	0.75	0.80	0.60	0.60	0.30	0.10	0.40	0.00	0.00	0.15	0.10
Foreign Equity	6.25%	16.0%	0.85	1.00	0.80	0.65	0.60	0.60	0.35	0.15	0.40	0.00	0.20	0.10	0.10
Emerging Markets	7.00%	19.0%	0.75	0.80	1.00	0.60	0.60	0.60	0.40	0.20	0.40	(0.10)	0.00	0.00	0.00
Private Equity	6.75%	20.0%	0.80	0.65	0.60	1.00	0.60	0.60	0.15	0.10	0.20	0.00	0.10	0.10	0.10
Absolute Return	5.00%	11.0%	0.60	0.60	0.60	0.60	1.00	0.70	0.10	0.10	0.20	0.20	0.20	0.20	0.10
High Yield	4.75%	14.0%	0.60	0.60	0.60	0.60	0.70	1.00	0.20	0.10	0.20	0.10	0.20	0.20	0.10
Commodities	4.50%	21.0%	0.30	0.35	0.40	0.15	0.10	0.20	1.00	0.10	0.00	0.10	0.10	0.40	0.00
Natural Resources	5.00%	10.0%	0.10	0.15	0.20	0.10	0.10	0.10	0.10	1.00	0.10	0.00	0.10	0.00	0.00
Real Estate	6.00%	15.0%	0.40	0.40	0.40	0.20	0.20	0.20	0.00	0.10	1.00	0.00	0.20	0.20	0.00
Domestic Bonds	1.75%	5.5%	0.00	0.00	(0.10)	0.00	0.20	0.10	0.10	0.00	0.00	1.00	0.70	0.60	0.30
Foreign Bonds	2.25%	5.8%	0.00	0.20	0.00	0.10	0.20	0.20	0.10	0.10	0.20	0.70	1.00	0.50	0.30
Inflation - Indexed	2.25%	5.1%	0.15	0.10	0.00	0.10	0.20	0.20	0.40	0.00	0.20	0.60	0.50	1.00	0.40
Cash	1.00%	3.5%	0.10	0.10	0.00	0.10	0.10	0.10	0.00	0.00	0.00	0.30	0.30	0.40	1.00

Source: Company documents.

[a]Standard deviation of returns.

Exhibit 18 Portfolio Allocation: Sixteen Different Portfolios Along the Efficient Frontier

Risk & Return Characteristics			Allocations to Different Asset Classes (%)												
Expect'd Real Ret. (%)	Std.Dev. (%)	Sharpe Ratio[a]	Dom. Equity	For. Equity	Emer. Markets	Private Equity	Abs. Return	High Yield	Comm.	Nat. Res.	Real Estate	Dom. Bonds	For. Bonds	Infl - Indx.	Cash
4.5	5.90	0.59	0	0	0	3.0	18.9	0	1.5	34.1	13.1	2.0	6.6	32.2	(11.5)
4.6	6.06	0.59	0	0	0	3.1	19.5	0	1.5	35.1	13.5	1.6	6.9	33.2	(14.3)
4.7	6.22	0.60	0	0	0	3.3	20.0	0	1.5	36.0	13.9	1.1	7.1	34.2	(17.2)
4.8	6.38	0.60	0	0	0	3.4	20.6	0	1.6	36.9	14.2	0.7	7.3	35.3	(20.0)
4.9	6.54	0.60	0	0	0	3.5	21.2	0	1.6	37.9	14.6	0.3	7.6	36.3	(22.9)
5.0	6.70	0.60	0	0	0	3.7	21.8	0	1.6	38.8	15.0	0	7.7	37.3	(25.7)
5.1	6.87	0.60	0	0	0	3.9	22.3	0	1.6	39.7	15.4	0	7.7	38.1	(28.6)
5.2	7.03	0.60	0	0	0	4.0	22.8	0	1.7	40.7	15.8	0	7.7	38.9	(31.5)
5.3	7.20	0.60	0	0	0	4.2	23.4	0	1.7	41.6	16.2	0	7.7	39.7	(34.5)
5.4	7.37	0.60	0	0	0	4.3	23.9	0	1.7	42.5	16.5	0	7.7	40.6	(37.4)
5.5	7.53	0.60	0	0	0	4.5	24.5	0	1.8	43.5	16.9	0	7.7	41.4	(40.3)
5.6	7.70	0.60	0	0	0	4.7	25.0	0	1.8	44.4	17.3	0	7.7	42.2	(43.2)
5.7	7.87	0.60	0	0	0	4.8	25.6	0	1.8	45.4	17.7	0	7.7	43.1	(46.1)
5.8	8.04	0.60	0	0	0	5.0	26.1	0	1.9	46.3	18.1	0	7.7	43.9	(49.0)
5.9	8.21	0.60	0	0	0	5.2	26.9	0	2.1	47.3	18.8	0	6.9	42.8	(50.0)
6.0	8.39	0.60	0	0	0	5.3	27.7	0	2.5	48.4	19.7	0	5.7	40.7	(50.0)

[a] Sharpe Efficiency Ratio = (Expected Return-Return on cash) / (Standard Deviation)

	Constraints (%)		
	Lower	Upper	Policy
Domestic Equity	0	100	11
Foreign Equity	0	100	11
Emerging Markets	0	100	11
Private Equity	0	100	13
Absolute Return	0	100	16
High Yield	0	100	2
Commodities	0	100	5
Natural Resources	0	100	9
Real Estate	0	100	9
Domestic Bonds	0	100	4
Foreign Bonds	0	100	2
Inflation - Indexed	0	100	5
Cash	(50)	100	2

Source: Company documents

Exhibit 19d Portfolio Allocation: Eight Different Portfolios Along the Efficient Frontier, But Constrained Near the Current Policy Portfolio

Risk & Return Characteristics			Allocations to Different Asset Classes (%)												
Expect'd Real Ret. (%)	Std. Dev. (%)	Sharpe Ratio[a]	Dom. Equity	For. Equity	Emer. Markets	Private Equity	Abs. Return	High Yield	Comm.	Nat. Res.	Real Estate	Dom. Bonds	For. Bonds	Infl - Indx.	Cash
4.6	6.38	0.56	1	1.0	1.0	5.3	26	(5.3)	5.2	19	19	8.9	12.0	15.0	(8)
4.8	6.81	0.56	1	1.6	1.0	7.5	26	(4.1)	6.1	19	19	3.9	12.0	15.0	(8)
5.0	7.29	0.55	1	3.4	1.7	8.7	26	(3.8)	6.5	19	19	(0.6)	12.0	15.0	(8)
5.2	7.80	0.54	1	5.0	2.7	9.9	26	(3.6)	6.9	19	19	(4.9)	12.0	15.0	(8)
5.4	8.35	0.53	1	6.9	4.4	11.4	26	(4.6)	6.7	19	19	(6.0)	9.2	15.0	(8)
5.6	8.93	0.52	1	9.0	5.4	12.7	26	(4.6)	7.1	19	19	(6.0)	6.0	13.4	(8)
5.8	9.53	0.50	1	10.9	6.4	14.1	26	(4.5)	7.6	19	19	(6.0)	3.9	10.5	(8)
6.0	10.15	0.49	1	12.9	7.4	15.4	26	(4.3)	8.2	19	19	(6.0)	1.9	7.6	(8)

[a]Sharpe Efficiency Ratio = (Expected Return - Return on cash) / (Standard Deviation)

	Constraints (%)		
	Lower	Upper	Policy
Domestic Equity	1	21	11
Foreign Equity	1	21	11
Emerging Markets	1	21	11
Private Equity	3	23	13
Absolute Return	6	26	16
High Yield	(8)	12	2
Commodities	(5)	15	5
Natural Resources	(1)	19	9
Real Estate	(1)	19	9
Domestic Bonds	(6)	14	4
Foreign Bonds	(8)	12	2
Inflation - Indexed	(5)	15	5
Cash	(8)	12	2

Source: Company documents.

The Need for Shortcut Approaches

- These difficulties have motivated a search for shortcut methods to find optimal portfolios:
- Capital Asset Pricing Model (CAPM).
- Multifactor models.

These models also have broader implications:

- Testable restrictions on asset returns.
- Capital budgeting (what discount rate to use in evaluating investment projects).
- Mutual fund performance evaluation (how large a return should one expect given the risk that a fund manager is taking).