Stochastic Discount Factor (1)

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Outline

- Complete markets
  - Unique stochastic discount factor (SDF)
  - Utility interpretation
  - Perfect risksharing and the representative agent
- Incomplete markets
  - Unique SDF in the space of payoffs
- Properties of the SDF
  - Risk premia
  - Volatility bounds
  - Factor structure

Discrete-State Model with Complete Markets

- Discrete-state model with states of nature $s = 1...S$.
- Contingent claim price $P_c(s)$ for \$1 in state s, \$0 otherwise.
- All contingent claims exist so markets are complete.
- Any other asset defined by payoffs $X(s)$ in state s, across s.

Law of One Price:

$$P(X) = \sum_{s=1}^{S} P_c(s)X(s).$$

The SDF in a Complete Market

$$P(X) = \sum_{s=1}^{S} P_c(s)X(s).$$

Multiply and divide by the probability of each state, $\pi(s)$:

$$P(X) = \sum_{s=1}^{S} \pi(s)\frac{P_c(s)}{\pi(s)}X(s) = \sum_{s=1}^{S} \pi(s)M(s)X(s) = \mathrm{E}[MX],$$

where $M(s)$ = stochastic discount factor (SDF).
For now, assume that agents all agree on these state probabilities.

Riskless Interest Rate

Riskless asset has $X(s) = 1$ in every state. The price

$$P_f = \sum_{s=1}^{S} P_c(s) = \sum_{s=1}^{S} \pi(s)\frac{P_c(s)}{\pi(s)} = \mathrm{E}[M],$$

so the riskless interest rate

$$1 + R_f = \frac{1}{P_f} = \frac{1}{\mathrm{E}[M]}.$$

Risk-Neutral Probabilities

$$\pi^*(s) = (1 + R_f)P_c(s) = \frac{M(s)}{\mathrm{E}[M]}\pi(s).$$

We have $\pi^*(s) > 0$ and $\sum_s \pi^*(s) = 1$, so they can be interpreted as if they were probabilities. We can rewrite the asset equation as

$$P(X) = \left(\frac{1}{1+R_f}\right)\sum_{s=1}^{S}\pi^*(s)X(s) = \left(\frac{1}{1+R_f}\right)\mathrm{E}^*[X].$$

The price of any asset is the pseudo-expectation of its payoff, discounted at the riskless interest rate.

Utility Maximization and the SDF

Consider an investor with initial wealth Y_0 and income $Y(s)$. The investor's maximization problem is

$$\text{Max } u(C_0) + \sum_{s=1}^{S} \beta\pi(s)u(C(s))$$

subject to

$$C_0 + \sum_{s=1}^{S} P_c(s)C(s) = Y_0 + \sum_{s=1}^{S} P_c(s)Y(s).$$

Utility Maximization and the SDF

First-order conditions

$$\begin{aligned} u'(C_0) &= \lambda \\ \beta\pi(s)u'(C(s)) &= \lambda P_c(s) \text{ for } s = 1...S. \end{aligned}$$

where λ is Lagrange multiplier on budget constraint. Thus

$$M(s) = \frac{P_c(s)}{\pi(s)} = \frac{\beta u'(C(s))}{u'(C_0)} = \frac{\beta u'(C(s))}{\lambda}$$

and

$$\frac{M(s_1)}{M(s_2)} = \frac{u'(C(s_1))}{u'(C(s_2))}.$$

The ratio of SDF realizations across states is the ratio of marginal utilities across states. (Assumption: Common beliefs!)

Perfect Risksharing

Since this is true for any two investors i and j, we also have

$$\frac{u_i'(C_{t+1}^i)}{u_i'(C_t^i)} = \frac{u_j'(C_{t+1}^j)}{u_j'(C_t^j)},$$

assuming a common time discount factor β.Condition holds ex post, not just ex ante, so is extremely strong: perfect risksharing.

This condition also characterizes the solution to the social planner's problem

$$\mathrm{Max}\ \lambda_i E \sum_t \beta^t u_i(C_t^i) + \lambda_j E \sum_t \beta^t u_j(C_t^j)$$

subject to $C_t^i + C_t^j = C_t$. Allocation of consumption is Pareto optimal.

The Martingale Method

The above logic has been applied to solve portfolio choice problems. In a model with only financial wealth and a single period,

$$C^j_{t+1} = X^j_{t+1},$$

where X^j_{t+1} is the payoff on investor j's portfolio. Given complete markets there is a unique SDF M_{t+1} such that

$$M_{t+1} = \frac{\beta}{\lambda_j} u'_j(X^j_{t+1}) \Longrightarrow X^j_{t+1} = u_j'^{-1}\left(\frac{\lambda_j}{\beta} M_{t+1}\right).$$

We solve for the λ_j that makes the payoff X^j_{t+1} affordable at time t given the investor's current wealth. Then the investor holds a portfolio of contingent claims that delivers X^j_{t+1} at time $t+1$. Cox and Huang (1989).

Complete Markets and the Representative Agent

In complete markets, all agents have the same ordering of marginal utility, and hence consumption, across states. So we can number states such that

$$C^i(s_1) \leq C^i(s_2) \leq ... \leq C^i(s_S)$$

for all agents i. Define aggregate consumption $C(s) = \sum_i C^i(s)$. Then we have

$$C(s_1) \leq C(s_2) \leq ... \leq C(s_S).$$

Also, we have

$$M(s_1) \geq M(s_2) \geq ... \geq M(s_S).$$

Complete Markets and the Representative Agent

Now find a function $g(C(s))$ s.t.

$$\frac{g(C(s_j))}{g(C(s_k))} = \frac{M(s_j)}{M(s_k)}$$

for all states j and k. The above ordering conditions ensure that this is always possible, with $g > 0$ and $g' \leq 0$. Finally, integrate to find a function $v(C(s))$ s.t.

$$v'(C(s)) = g(C(s)).$$

The function $v(.)$ is the utility function of a representative agent who consumes aggregate consumption and holds the market portfolio of all wealth.

Market portfolio is efficient (we can find a concave utility function that prefers it).

But representative agent preferences need not relate to individual preferences ("mongrel aggregation").

The SDF in Incomplete Markets

What if markets are incomplete? We continue to observe a set of payoffs X and prices P. The set of all payoffs (the payoff space) is Ξ. We assume:

(A1) Portfolio formation $X_1, X_2 \in \Xi \Longrightarrow aX_1 + bX_2 \in \Xi$ for any real a, b.

(A2) Law of One Price $P(aX_1 + bX_2) = aP(X_1) + bP(X_2)$.

Theorem. A1, A2 $\Longrightarrow$ there exists a unique payoff $X^* \in \Xi$ s.t. $P(X) = E(X^*X)$ for all $X \in \Xi$.

The SDF in Incomplete Markets

Theorem. A1, A2 $\Longrightarrow$ *there exists a unique payoff* $X^* \in \Xi$ *s.t.* $P(X) = E(X^*X)$ *for all* $X \in \Xi$.

Sketch of proof: Assume that there are N basis payoffs $X_1, ..., X_N$. Construct a vector $X = [X_1 ... X_N]'$. Write the set $\Xi = \{c'X\}$. We want to find some vector $X^* = c'X$ that prices the basis payoffs. That is, we want

$$P = \mathrm{E}[X^*X] = \mathrm{E}[XX'c]$$

which requires

$$c = \mathrm{E}[XX']^{-1}P$$

and

$$X^* = P'\mathrm{E}[XX']^{-1}X.$$

This construction for X^* always exists and unique provided that the matrix $\mathrm{E}[XX']$ is nonsingular.

The SDF in Incomplete Markets

- We can subtract means and rewrite all of this in terms of covariance matrices.
- Only the SDF that is a linear combination of asset payoffs is unique. There may be many other SDF's of the form $M = X^* + \epsilon$, where $\mathrm{E}[\varepsilon X] = 0$. These must all have higher variance than X^* (Hansen-Jagannathan variance bound).
- X^* is the projection of every SDF onto the space of payoffs. Thus it can be thought of as the portfolio of assets that best mimics the behavior of every SDF.

The SDF in Incomplete Markets

Definition. *A payoff space* Ξ *and pricing function* $P(X)$ *have absence of arbitrage if all* X *s.t.* $X \geq 0$ *always and s.t.* $X > 0$ *with positive probability have* $P(X) > 0$.

Theorem. $P = E(MX)$ *and* $M(s) > 0 \Longrightarrow$ *absence of arbitrage.*
Proof: $P(X) = \sum_s \pi(s)M(s)X(s)$, and no terms in this expression are ever negative.

Theorem. *Absence of arbitrage* $\Longrightarrow \exists M$ *s.t.* $P = E(MX)$ *and* $M(s) > 0$.
Proof: See Cochrane, *Asset Pricing*, Chapter 4, for a geometric proof. The intuition is that with absence of arbitrage, we can always find a complete-markets, contingent-claims economy (in general, many such economies) that could have generated the asset prices we observe.

The SDF and Risk Premia

For a general risky asset i, we have

$$
\begin{aligned}
P_{it} &= \mathrm{E}_t[M_{t+1}X_{i,t+1}] = \mathrm{E}_t[M_{t+1}]\mathrm{E}_t[X_{i,t+1}] + \mathrm{Cov}_t(M_{t+1}, X_{i,t+1}) \\
&= \frac{\mathrm{E}_t[X_{i,t+1}]}{(1+R_{f,t+1})} + \mathrm{Cov}_t(M_{t+1}, X_{i,t+1}).
\end{aligned}
$$

The SDF and Risk Premia

For assets with positive prices, we can divide through by P_{it} and use $(1 + R_{i,t+1}) = X_{i,t+1}/P_{it}$ to get

$$1 = \mathrm{E}_t[M_{t+1}(1 + R_{i,t+1})] = \mathrm{E}_t[M_{t+1}]\mathrm{E}_t[1 + R_{i,t+1}] + \mathrm{Cov}_t(M_{t+1}, R_{i,t+1})$$

$$\mathrm{E}_t[1 + R_{i,t+1}] = (1 + R_{f,t+1})(1 - \mathrm{Cov}_t(M_{t+1}, R_{i,t+1})).$$

$$\mathrm{E}_t(R_{i,t+1} - R_{f,t+1}) = \frac{-\mathrm{Cov}_t(M_{t+1}, R_{i,t+1} - R_{f,t+1})}{\mathrm{E}_t M_{t+1}}.$$

The SDF and Risk Premia (Lognormal Version)

Assume joint lognormality of asset returns and the SDF. Log riskless rate is

$$r_{f,t+1} = -\mathrm{E}_t m_{t+1} - \sigma^2_{mt}/2,$$

where $r_{f,t+1} \equiv \log(1 + R_{f,t+1})$, $m_{t+1} \equiv \log(M_{t+1})$, and $\sigma^2_{mt} = \mathrm{Var}_t(m_{t+1})$.

Log risk premium with Jensen's Inequality correction is

$$\mathrm{E}_t r_{i,t+1} - r_{f,t+1} + \sigma_i^2/2 = -\sigma_{imt},$$

where $\sigma_{imt} \equiv Cov_t(r_{i,t+1}, m_{t+1})$.

Volatility Bounds on the SDF

Shiller (1982) considers a single risky asset:

$$\begin{aligned} \mathrm{E}_t(R_{i,t+1} - R_{f,t+1}) &= \frac{-\mathrm{Cov}_t(M_{t+1}, R_{i,t+1} - R_{f,t+1})}{\mathrm{E}_t M_{t+1}} \\ &\leq \frac{\sigma_t(M_{t+1})\sigma_t(R_{i,t+1} - R_{f,t+1})}{\mathrm{E}_t M_{t+1}}. \end{aligned}$$

$$\frac{\sigma_t(M_{t+1})}{\mathrm{E}_t M_{t+1}} \geq \frac{\mathrm{E}_t(R_{i,t+1} - R_{f,t+1})}{\sigma_t(R_{i,t+1} - R_{f,t+1})}.$$

Log version, assuming joint lognormality:

$$\sigma_{mt} \geq \frac{\mathrm{E}_t r_{i,t+1} - r_{f,t+1} + \sigma_i^2/2}{\sigma_{it}}.$$

Simple way to understand the equity premium puzzle.

Entropy and Cumulants

Alvarez-Jermann (2005), Backus-Chernov-Martin (2009). Define entropy as

$$L(\widetilde{X}) = \log \mathrm{E}\widetilde{X} - \mathrm{E}\log(\widetilde{X}) \geq 0.$$

For a constant a, $L(\widetilde{aX}) = L(\widetilde{X})$.

The cumulant-generating function of random variable x is

$$k(s;x) = \log \mathrm{E}[\exp(sx)] = \sum_{j=1}^{\infty} \frac{\kappa_j(x)s^j}{j!},$$

where the cumulants $\kappa_j(x)$ are: $\kappa_1 =$ mean, $\kappa_2 =$ standard deviation, $\kappa_3/\kappa_2^{3/2} =$ skewness, $\kappa_4/\kappa_2^2 =$ excess kurtosis, etc.

$$L(\widetilde{X}) = k(1;x) - \kappa_1(x) = \sum_{j=2}^{\infty} \frac{\kappa_j(x)}{j!}.$$

Entropy Bound on the SDF

In a finite-state model, we have

$$M(s) = P_f \frac{\pi^*(s)}{\pi(s)}.$$

If returns are iid, P_f is constant, so

$$L(M) = L\left(\frac{\pi^*}{\pi}\right) = \log \mathrm{E}\left(\frac{\pi^*}{\pi}\right) - \mathrm{E}\log\left(\frac{\pi^*}{\pi}\right) = -\mathrm{E}\log\left(\frac{\pi^*}{\pi}\right).$$

The entropy of the SDF is then a measure of the deviation of π^* from π. Alvarez and Jermann (2005) show that

$$L(M) \geq E[r_j - r_f].$$

A high log risk premium implies high entropy of the SDF, but this may be due to higher moments rather than high variance of log SDF. ("Rare disasters" literature.)

Entropy Bound on the SDF: Proof

1. Since $\mathrm{E}[M(1+R_j)] = 1$, $\mathrm{E}m + \mathrm{E}r_j \leq \log \mathrm{E}[M(1+R_j)] = 0$. This implies

$$\mathrm{E}r_j \leq -\mathrm{E}m.$$

The weak inequality becomes an equality for the growth-optimal portfolio.
2. Allow for time-variation in the price of a riskless asset: $P_{1t} = \mathrm{E}_t M_{t+1}$. The entropy of the riskless asset price is

$$L(P_1) = \log \mathrm{E}P_1 - \mathrm{E}p_1 = \log \mathrm{E}M + \mathrm{E}r_1.$$

3. Putting these together,

$$\begin{aligned} L(M) &= \log \mathrm{E}M - \mathrm{E}m \\ &\geq \log \mathrm{E}M + \mathrm{E}r_j \\ &= L(P_1) + \mathrm{E}(r_j - r_1) \\ &\geq \mathrm{E}(r_j - r_1). \end{aligned}$$

Hansen-Jagannathan Bounds

Hansen-Jagannathan (1991) extended Shiller volatility bound to multiple risky assets.

Suppose there are N risky assets and no riskless asset, so the mean of the SDF is not pinned down by the mean return on any asset. Write this unknown mean SDF as $\overline{M}$. The minimum-variance stochastic discount factor is a linear combination of asset returns:

$$M_t^*(\overline{M}) = \overline{M} + (R_t - R)'\beta(\overline{M})$$

for some coefficient vector $\beta(\overline{M})$. Any other SDF has a higher variance.

Hansen-Jagannathan Bounds

H-J use the fundamental equation of asset pricing,

$$\iota = E[(\iota + R_t)M_t],$$

to show that

$$\mathrm{Var}(M_t^*(\overline{M})) = A\overline{M}^2 - 2B\overline{M} + C,$$

where $A = (\iota + \overline{R})'\Sigma^{-1}(\iota + \overline{R})$, $B = \iota'\Sigma^{-1}(\iota + \overline{R})$, and $C = \iota'\Sigma^{-1}\iota$ are just as we defined them in the standard mean-variance analysis, except with gross mean returns. Σ is the variance-covariance matrix of asset returns.

The Benchmark Return

If we augment the set of risky asset returns with a hypothetical riskless return $1/\overline{M}$, then we can define a benchmark return

$$1 + R_{bt}(\overline{M}) = \frac{M_t^*(\overline{M})}{E[M_t^*(\overline{M})^2]}.$$

The benchmark return has the following properties:

- It lies on the minimum-variance frontier (the lower part, not the mean-variance efficient frontier).
- It has the highest possible correlation with the SDF.
- Beta pricing works with the benchmark return:

$$\frac{1/\overline{M} - (1 + \overline{R}_b)}{\sigma_b} \leq \frac{\sigma_M(\overline{M})}{\overline{M}}.$$

Elegant geometrical interpretation.

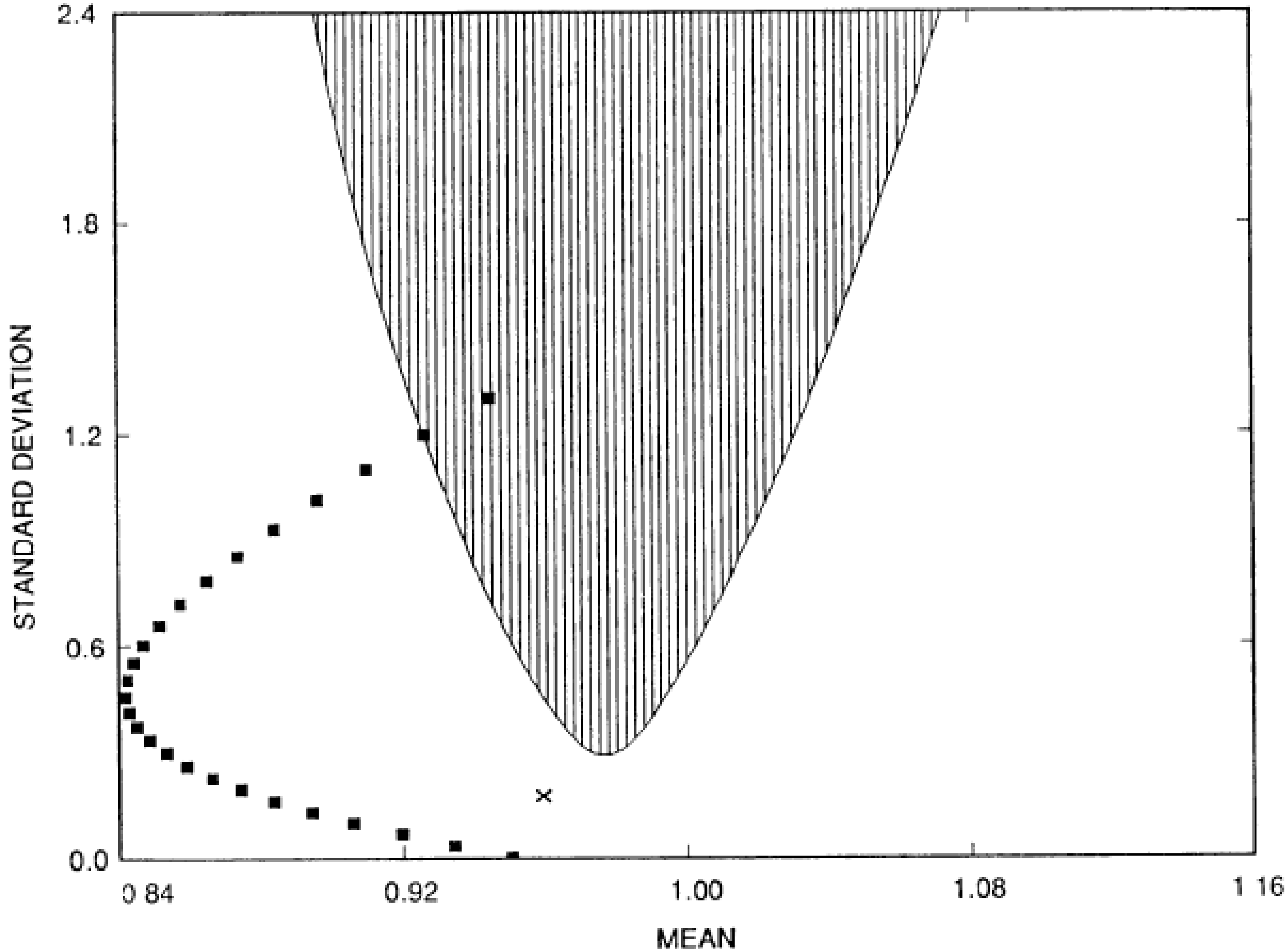

FIG. 1.—IMRS frontier computed using annual data

Hansen and Jagannathan, JPE 1991

Factor Structure of the SDF

Assume that the SDF is a linear combination of K common factors $f_{k,t+1}$, $k = 1...K$. For simplicity assume that the factors have conditional mean zero and are orthogonal to one another. If

$$M_{t+1} = a_t - \sum_{k=1}^{K} b_{kt} f_{k,t+1},$$

then

$$\begin{aligned} -\mathrm{Cov}_t(M_{t+1}, R_{i,t+1} - R_{f,t+1}) &= \sum_{k=1}^{K} b_{kt}\sigma_{ikt} \\ &= \sum_{k=1}^{K} (b_{kt}\sigma_{kt}^2)\left(\frac{\sigma_{ikt}}{\sigma_{kt}^2}\right) = \sum_{k=1}^{K} \lambda_{kt}\beta_{ikt}. \end{aligned}$$

Factor Structure of the SDF

Note how this is consistent with earlier insights about multifactor models:

- Single-period model with quadratic utility implies consumption equals wealth and marginal utility is linear. Thus the SDF must be linear in future wealth, or equivalently the market portfolio return.
- In a single-period model with K common shocks and completely diversifiable idiosyncratic risk, marginal utility and hence the SDF can depend only on the common shocks.